8.3 - Nonhomogeneous Linear Systems

$\boxed{\mathbf{X}' = \mathbf{A}\mathbf{X} +} \mathbf{F}(t)$ is a nonhomogeneous system if $\mathbf{F}(t) \neq \mathbf{0}$.

8.2

Example: Use the method of undetermined coefficients to solve the given nonhomogeneous system. → Homog.

$$\begin{vmatrix} 1-\lambda & -4 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 2\lambda + 1 = -17 + 1$$

$$\mathbf{X}' = \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 4t + 9e^{6t} \\ -t + e^{6t} \end{pmatrix}$$

Find $\vec{X}_c$ (from 8.2), the solution to the associated homogeneous system.

Here, $\lambda = 1 \pm 4i$, so we have $\begin{pmatrix} -4i & -4 \\ 4 & -4i \end{pmatrix} \vec{K} = \vec{0}$

R_1: $-ik_1 - k_2 = 0 \Rightarrow k_1 = ik_2$ $\quad \vec{B}_1 \quad \vec{B}_2$

Let $k_2 = -i \Rightarrow k_1 = 1 \Rightarrow \vec{K} = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + \underline{\begin{pmatrix} 0 \\ -1 \end{pmatrix}} i$

$$\vec{X}_c = c_1 e^t \left[\underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \cos 4t - \underline{\begin{pmatrix} 0 \\ -1 \end{pmatrix}} \sin 4t \right] + c_2 e^t \left[\underline{\begin{pmatrix} 0 \\ -1 \end{pmatrix}} \cos 4t + \underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \sin 4t \right]$$

$$\vec{F}(t) = \begin{pmatrix} 4t + 9e^{6t} \\ -t + e^{6t} \end{pmatrix} = \underbrace{\begin{pmatrix} 4 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\text{Form } \vec{A}t + \vec{B}} + \begin{pmatrix} 9 \\ 1 \end{pmatrix} e^{6t}$$

↑

$$\vec{X}_p = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} t + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} e^{6t}$$ goes into

$$\vec{X}' = A\vec{X} + \vec{F}$$

$$\begin{pmatrix} a_3 \\ b_3 \end{pmatrix} + \begin{pmatrix} 6a_1 \\ 6b_1 \end{pmatrix} e^{6t} = \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} a_3 t + a_2 + a_1 e^{6t} \\ b_3 t + b_2 + b_1 e^{6t} \end{pmatrix} + \begin{pmatrix} 4t + 9e^{6t} \\ -t + e^{6t} \end{pmatrix}$$

$$\begin{pmatrix} a_3 + 6a_1e^{6t} \\ b_3 + 6b_1e^{6t}\end{pmatrix} = \begin{pmatrix}(a_3-4b_3+4)t + (a_2-4b_2) + (a_1-4b_1+9)e^{6t} \\ (4a_3+b_3-1)t + (4a_2+b_2) + (4a_1+b_1+1)e^{6t}\end{pmatrix}$$

t: $a_3-4b_3+4=0$ Const: $a_3 = a_2-4b_2$ exp: $6a_1 = a_1-4b_1+9$

$4a_3+b_3-1=0$ $b_3 = 4a_2+b_2$ $6b_1 = 4a_1+b_1+1$

AALOW:

$$\boxed{\vec{x} = c_1e^{t}\begin{pmatrix}\cos 4t\\ \sin 4t\end{pmatrix} + c_2e^{t}\begin{pmatrix}\sin 4t\\ -\cos 4t\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}t + \begin{pmatrix}4/17\\1/17\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}e^{6t}}$$

this is of particularly limited use.

$\vec{X}' = A\vec{X} + \vec{F}$ before: $y_p = u_1y_1 + u_2y_2$

Variation of parameters

The complementary solution is $\mathbf{X}_c(t) = \mathbf{\Phi}(t)\mathbf{C}$.

Definition: $\Phi(t)$ is the **fundamental matrix** of the system. $\det(\Phi)$

For a system with independent solutions, the Wronskian is nonzero, so $\mathbf{\Phi}^{-1}$ exists. Since Φ is a solution to $\mathbf{X}' = \mathbf{AX}$, we have $\mathbf{\Phi}' = \mathbf{A\Phi}$. Suppose $\mathbf{X}_p = \mathbf{\Phi}(t)\mathbf{U}(t)$, where $\mathbf{U}(t)$ is a column matrix of unknown functions.

2×1 2×2 2×1

Recall that order matters for matrix products

$\vec{X}_p' = \Phi'\vec{u} + \Phi\vec{u}'$ into $\vec{X}' = A\vec{X} + \vec{F}$

gives $\Phi'\vec{u} + \Phi\vec{u}' = A\Phi\vec{u} + \vec{F}$ but

$\cancel{A\Phi\vec{u}} + \Phi\vec{u}' = \cancel{A\Phi\vec{u}} + \vec{F}$

$\frac{1}{3}3x = \frac{1}{3}15$ $\Phi^{-1}\Phi\vec{u}' = \Phi^{-1}\vec{F} \Rightarrow \vec{u}' = \Phi^{-1}\vec{F}$

$1x = 5$

$x = 5$ Matrix inverse: $AA^{-1} = A^{-1}A = I = \begin{bmatrix}1 & 0\\ 0 & \ddots 1\end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \vec{u} = \int \Phi^{-1}\vec{F}\,dt$$

$\vec{X}_p = \Phi\vec{u}$, so we get

$$\mathbf{X}(t) = \underbrace{\mathbf{\Phi}(t)\mathbf{C}}_{\vec{X}_c} + \underbrace{\mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,dt}_{\vec{X}_p}$$

Example: Use variation of parameters to solve the given nonhomogeneous system.

$$\frac{dx}{dt} = 2x - y$$

$$\frac{dy}{dt} = 3x - 2y + 4t$$

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

$\lambda = 1$:

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\vec{K} = \vec{0} \Rightarrow \vec{K}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda = -1$:

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\vec{K} = \vec{0} \Rightarrow \vec{K}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\vec{X}_c = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t} = \begin{pmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Phi = \begin{pmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{pmatrix}$$

Aside: Matrix inverses. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A^{-1}A = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{pmatrix}$$

$$\Phi^{-1} = \frac{1}{2}\begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{pmatrix}$$

$$\vec{F} = \begin{pmatrix} 0 \\ 4t \end{pmatrix}$$

$$\vec{u}' = \Phi^{-1}\vec{F} = \frac{1}{2}\begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{pmatrix}\begin{pmatrix} 0 \\ 4t \end{pmatrix} = \begin{pmatrix} -4te^{-t} \\ 4te^{t} \end{pmatrix}$$

$$\vec{u} = \frac{1}{2}\int \begin{pmatrix} -4te^{-t} \\ 4te^{t} \end{pmatrix} dt \Rightarrow \vec{u} = \begin{pmatrix} 2te^{-t} + 2e^{-t} \\ 2te^{t} - 2e^{t} \end{pmatrix} + \cancel{\vec{c}}$$

$$\vec{X}_p = \Phi\vec{u} = \begin{pmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{pmatrix}\begin{pmatrix} 2te^{-t} + 2e^{-t} \\ 2te^{t} - 2e^{t} \end{pmatrix}$$

$$= \begin{pmatrix} 4t \\ 8t - 4 \end{pmatrix}$$

$$\vec{X} = \vec{X}_c + \vec{X}_p = \begin{pmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{pmatrix}\vec{C} + \begin{pmatrix} 4t \\ 8t - 4 \end{pmatrix}$$

$$\boxed{\vec{X}(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t} + \begin{pmatrix} 4 \\ 8 \end{pmatrix}t + \begin{pmatrix} 0 \\ -4 \end{pmatrix}}$$

WA $c_1\langle 1, 1\rangle e^{t} \;\ldots$

Example: Use variation of parameters to solve the given nonhomogeneous system.

$$\mathbf{X}' = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix} e^{2t}$$

$$\vec{X}_c = e^{2t}\left\{c_1\left[\begin{pmatrix}1\\0\end{pmatrix}\cos 2t - \begin{pmatrix}0\\-2\end{pmatrix}\sin 2t\right] + c_2\left[\begin{pmatrix}0\\-2\end{pmatrix}\cos 2t + \begin{pmatrix}1\\0\end{pmatrix}\sin 2t\right]\right\}$$

$$\vec{X}_c = \underset{\Phi}{\boxed{\begin{pmatrix} e^{2t}\cos 2t & e^{2t}\sin 2t \\ 2e^{2t}\sin 2t & -2e^{2t}\cos 2t \end{pmatrix}}}\vec{C}$$

$$\vec{X}_p = \Phi\vec{U} = \Phi\int \Phi^{-1}\vec{F}\,dt$$

$$\Phi^{-1} = \frac{e^{2t}}{-2e^{4t}\cos^2 2t - 2e^{4t}\sin^2 2t}\begin{pmatrix} -2\cos 2t & -\sin 2t \\ -2\sin 2t & \cos 2t \end{pmatrix}$$

$$-2e^{4t}(\cos^2 2t + \sin^2 2t)$$

$$\Phi^{-1} = \frac{1}{-2e^{2t}}\begin{pmatrix} -2\cos 2t & -\sin 2t \\ -2\sin 2t & \cos 2t \end{pmatrix}$$

$$\Phi^{-1} = e^{-2t}\begin{pmatrix} \cos 2t & \frac{1}{2}\sin 2t \\ \sin 2t & -\frac{1}{2}\cos 2t \end{pmatrix}$$

$$\Phi^{-1}\vec{F} = e^{-2t}\begin{pmatrix} \cos 2t & \frac{1}{2}\sin 2t \\ \sin 2t & -\frac{1}{2}\cos 2t \end{pmatrix}\begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}e^{2t}$$

$$= \begin{pmatrix} \cos 2t\sin 2t + \cos 2t\sin 2t \\ \sin^2 2t - \cos^2 2t \end{pmatrix} = \begin{pmatrix} \sin 4t \\ -\cos 4t \end{pmatrix}$$

$$\int \Phi^{-1}\vec{F}\,dt = \begin{pmatrix} -\frac{1}{4}\cos 4t \\ -\frac{1}{4}\sin 4t \end{pmatrix}$$

$$\Phi\int \Phi^{-1}\vec{F}\,dt = e^{2t}\begin{pmatrix} \sin 2t & \cos 2t \\ 2\cos 2t & 2\sin 2t \end{pmatrix}\begin{pmatrix} -\frac{1}{4}\cos 4t \\ -\frac{1}{4}\sin 4t \end{pmatrix}$$

$-\frac{1}{4}(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$

$$\vec{X}_p = e^{2t}\begin{pmatrix} -\frac{1}{4}\sin 2t\cos 4t - \frac{1}{4}\cos 2t\sin 4t \\ -\frac{1}{2}\cos 2t\cos 4t - \frac{1}{2}\sin 2t\sin 4t \end{pmatrix}$$

~~$\vec{X} = \vec{X}_c + \vec{X}_p$ as before.~~

Alternative for inverse matrices:

$$(A \mid I) \to (I \mid A^{-1})$$

$\vec{X}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

For an initial-value problem of the form $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$, $X(t_0) = \mathbf{X}_0$, we have $\mathbf{X} = \mathbf{\Phi C} + \mathbf{\Phi}\int_{t_0}^{t}\mathbf{\Phi}^{-1}(s)\mathbf{F}(s)\,ds \implies \mathbf{X}(t_0) = \mathbf{\Phi}(t_0)\mathbf{C} \implies \mathbf{C} = \mathbf{\Phi}^{-1}(t_0)\mathbf{X}(t_0)$. Thus,

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{X}(t_0) + \mathbf{\Phi}(t)\int_{t_0}^{t}\mathbf{\Phi}^{-1}(s)\mathbf{F}(s)\,ds$$

Example: Solve the given initial-value problem.

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1/t \\ 1/t \end{pmatrix}, \quad \mathbf{X}(1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = 0 \Rightarrow \lambda = 0, 0$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \vec{K} = \vec{0} \Rightarrow \vec{K} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \vec{P} = \vec{K} \Rightarrow \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 1 \end{array}\right) \rightarrow p_1 - p_2 = 1$$

Let $p_2 = 0 \Rightarrow p_1 = 1 \quad \vec{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\vec{X}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

$$c_1 \vec{K} e^{\lambda t} + c_2 \left(\vec{K} t e^{\lambda t} + \vec{P} e^{\lambda t} \right)$$

$$\Phi = \begin{pmatrix} 1 & t+1 \\ 1 & t \end{pmatrix} \Rightarrow \Phi^{-1} = \frac{1}{-1} \begin{pmatrix} t & -t-1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -t & t+1 \\ 1 & -1 \end{pmatrix}$$

$$\Phi^{-1} \vec{F} = \begin{pmatrix} -t & t+1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{t} \\ \frac{1}{t} \end{pmatrix} = \begin{pmatrix} 1/t \\ 0 \end{pmatrix}$$

$$\int_1^t \begin{pmatrix} 1/s \\ 0 \end{pmatrix} ds = \begin{pmatrix} \ln t - \ln 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \ln t \\ 0 \end{pmatrix}$$

$$\Phi \begin{pmatrix} \ln t \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & t+1 \\ 1 & t \end{pmatrix} \begin{pmatrix} \ln t \\ 0 \end{pmatrix} = \begin{pmatrix} \ln t \\ \ln t \end{pmatrix}$$

$$\Phi(t) = \begin{pmatrix} 1 & t+1 \\ 1 & t \end{pmatrix}, \underbrace{\Phi^{-1}(1) = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}, \quad \vec{X}(1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}}^{\text{given:}}$$

$$\Phi^{-1}(1)\vec{X}(1) = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \vec{C}$$

$$\vec{X}(t) = \begin{pmatrix} 1 & t+1 \\ 1 & t \end{pmatrix} \begin{pmatrix} -4 \\ 3 \end{pmatrix} + \begin{pmatrix} \ln t \\ \ln t \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 3t-1 \\ 3t-4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ln t}$$

$$(AB)C = A(BC)$$

Adjoint method for A^{-1}:

$$\begin{pmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ \overset{-}{d} & \overset{+}{e} & \overset{-}{f} \\ g & h & i \end{pmatrix}$$

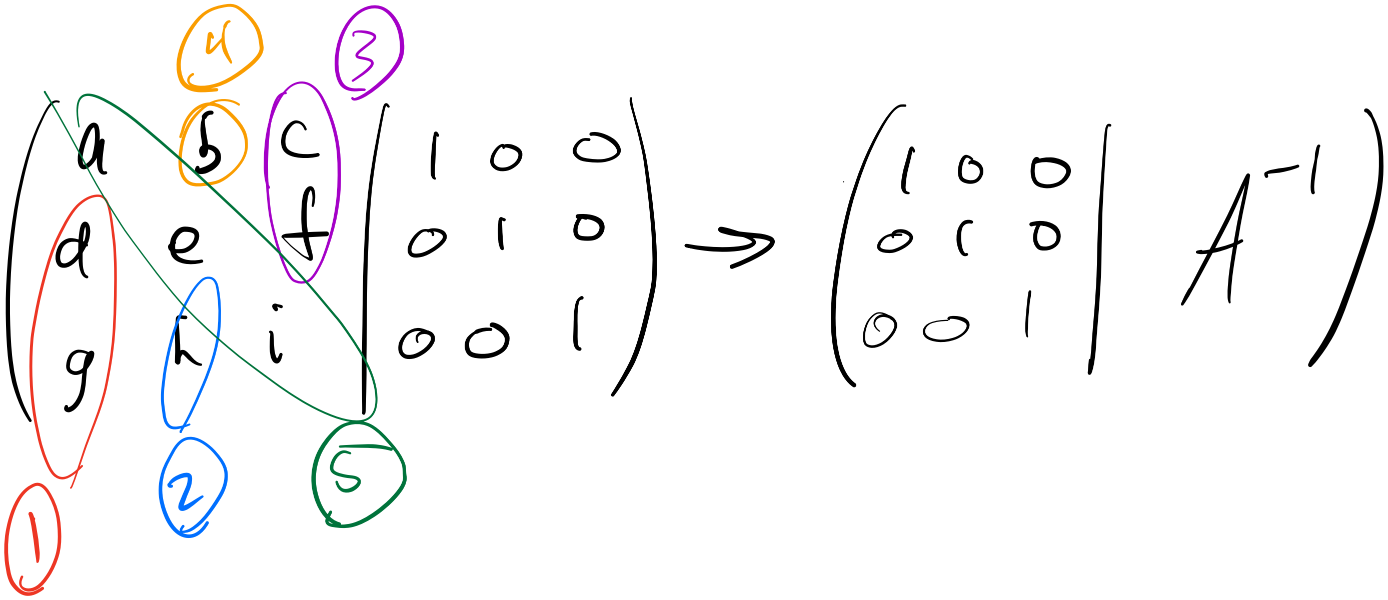
$$M_{11} = \begin{vmatrix} e & f \\ h & i \end{vmatrix} \qquad M_{12} = \begin{vmatrix} d & f \\ g & i \end{vmatrix} \qquad \cdots$$

$$C_{\underline{12}} = (-1)^{1+2} M_{12}$$

Matrix of cofactors: $\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$

Adjoint: $\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \operatorname{adj}(A)$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$



$$\left(\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right)$$

$$\begin{pmatrix} \overset{+}{2} & \overset{-}{5} & \overset{+}{-3} \\ \overset{-}{4} & \overset{+}{-7} & \overset{-}{8} \\ \overset{+}{5} & \overset{-}{1} & \overset{+}{6} \end{pmatrix}$$

$$M_{11} = \begin{vmatrix} -7 & 8 \\ 1 & 6 \end{vmatrix} \qquad M_{12} = \begin{vmatrix} 4 & 8 \\ 5 & 6 \end{vmatrix}$$

$$= -42 - 8 \qquad = 24 - 40$$

$$= -50 \qquad = -16$$

$$C_{11} = (-1)^{1+1}(-50) \qquad C_{12} = (-1)^{1+2}(-16)$$

$$C_{11} = -50 \qquad = 16$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

$\det(A) = ad - bc$

$M_{11} = d \qquad M_{12} = c \qquad C_{11} = d \qquad C_{12} = -c$

$\rightarrow$

$M_{21} = b \qquad M_{22} = a \qquad C_{21} = -b \qquad C_{22} = a$

cofactor matrix: $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

$\operatorname{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$